

# Can dual gravity be reconciled with $E_{11}$ ?

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## ABSTRACT

We extend a recently proposed formulation of dual gravity to the case of eleven-dimensional supergravity. The supersymmetric action corresponding to this alternative formulation is given, and it is shown that it leads to a set of first-order duality relations from which all second-order equations of motion follow as integrability conditions. On top of the fields corresponding to the conjectured  $E_{11}$  symmetry the action features Stückelberg gauge fields that facilitate the realization of arbitrary symmetries on the dual graviton. However, there is no gauge-fixing that allows to eliminate the Stückelberg fields. Therefore,  $E_{11}$  by itself is not a symmetry of eleven-dimensional supergravity but has to be extended at least by the Stückelberg symmetries.

# 1 Introduction

In recent years there has been increasing interest in a possible relation between supergravity theories and Kac-Moody algebras. As the most prominent example, it has been conjectured that the infinite-dimensional Kac-Moody algebra  $E_{11}$  is a symmetry of 11-dimensional supergravity and its lower-dimensional descendants [1, 2]. The main evidence for this conjecture is given by the observation that the level decompositions of  $E_{11}$  with respect to its finite-dimensional subalgebras reproduce the same p-form representations as expected for maximal supergravity when formulated in a democratic way that introduces for each p-form also its dual. For instance, the level decomposition of  $E_{11}$  with respect to  $SL(11)$  reproduces a 3-form at level 1, in agreement with the 3-form potential of 11-dimensional supergravity, but also a 6-form at level 2. Though 11-dimensional supergravity cannot be formulated entirely in terms of a dual 6-form, it is possible to encode the dynamics in a democratic way in terms of a first-order duality relation between the curvatures of the 3- and 6-form. Moreover, due to this appearance of extra gauge potentials the gauge symmetry is enhanced in such a way that it permits a global subgroup that, in turn, is in precise agreement with a certain positive-level truncation of a non-linear realization of  $E_{11}$ . To be specific,  $E_{11}$  predicts the following global symmetry on the 6-form,

$$\delta A_{\mu_1 \dots \mu_6} = \Lambda_{\mu_1 \dots \mu_6} + 20 \Lambda_{[\mu_1 \dots \mu_3} A_{\mu_4 \dots \mu_6]} , \quad (1.1)$$

while on the supergravity side there is a corresponding gauge symmetry with parameters  $\Lambda_{(2)}$  and  $\Lambda_{(5)}$ . The latter reduces to the global symmetry (1.1) upon specifying the parameters to linear space-time dependence,  $\Lambda_{\mu_1 \dots \mu_5}(x) = \Lambda_{\rho \mu_1 \dots \mu_5} x^\rho$ , etc. Thus, there is a reformulation of the dynamics of the p-form sector featuring the following two related properties:

- (i) the second-order field equations result as integrability conditions from first-order duality relations between the physical fields and their duals predicted by  $E_{11}$ ,
- (ii) the gauge symmetry is enhanced in agreement with  $E_{11}$ .

This correspondence between Kac-Moody algebras and (ungauged) supergravities naturally extends to lower dimensions and fewer numbers of supercharges. However, if one goes beyond low levels or the pure p-form sector, the situation becomes more subtle. In the  $D = 11$  decomposition, for instance, one finds a mixed-Young tableaux representation at level 3, which is interpreted as the dual of the graviton. Even though it is possible to formulate Einstein's theory in the linearization about flat space entirely in terms of a dual graviton [1, 3, 4], this turns out to be impossible for the non-linear theory [5]. Also, a formulation in terms of a duality relation is impossible even for linearized gravity once matter fields are incorporated [6]. Finally, there is no canonical way to associate to the rigid  $E_{11}$  symmetry on the dual graviton  $C_{(8,1)}$ , which is given by

$$\begin{aligned} \delta C_{\mu_1 \dots \mu_8, \nu} &= \xi_{\mu_1 \dots \mu_8, \nu} + 14 \Lambda_{\langle \mu_1 \dots \mu_6} A_{\mu_7 \mu_8 \nu \rangle} - 14 A_{\langle \mu_1 \dots \mu_6} \Lambda_{\mu_7 \mu_8 \nu \rangle} \\ &\quad + \frac{7}{9} \Lambda_{\langle \mu_1 \dots \mu_3} A_{\mu_4 \dots \mu_6} A_{\mu_7 \mu_8 \nu \rangle} , \end{aligned} \quad (1.2)$$

a local gauge symmetry in supergravity. More precisely, due to the non-trivial Young projection (here indicated by brackets  $\langle \rangle$ ) the global symmetry parameter cannot be identified with the curl of a local parameter in a way that would allow the definition of first-order curvatures.

Recently, a proposal has been made to overcome the problem, implied by the no-go theorems of [5], of finding an equivalent reformulation of (super-)gravity that contains the dual graviton and is valid at the non-linear level as well. Inspired by a similar approach to gauged supergravity (see [7] and references therein), an action has been given that contains the original metric via a topological term and an additional shift gauge field [8]. Moreover, in this formulation the non-linear Einstein equations can be encoded in a set of *two* duality relations, thereby resolving the problems mentioned above and preserving feature (i). This reformulation can be investigated quite independently of  $E_{11}$  and might be useful for other applications as well.

The aim of the present letter is two-fold. First, in section 2, we extend the proposed reformulation of [8] to the special case of 11-dimensional supergravity. In particular, we will show how all the symmetries of 11-dimensional supergravity, like supersymmetry, are realized. This reformulation preserves feature (i). Next, in section 3, we address the question whether feature (ii) is also preserved — being a priori independent of (i) —, namely whether this reformulation of 11-dimensional supergravity realizes the symmetries of  $E_{11}$ , in particular the one of the dual graviton given in eq. (1.2).

## 2 An alternative formulation of 11-dimensional supergravity

In order to present the alternative formulation of 11-dimensional supergravity, it is instructive to first show how the dual 6-form potential is introduced. After that we will introduce the dual graviton and, finally, we will discuss all the gauge symmetries of the alternative formulation, including supersymmetry.

### 2.1 Democratic formulation with 6-form potential

We start by giving a reformulation of 11-dimensional supergravity containing besides the standard fields, i.e., the metric, the 3-form  $A_{(3)}$  and the gravitino  $\psi_\mu$ , also the dual 6-form potential  $A_{(6)}$ . It turns out that this is possible provided one introduces in addition a 7-form gauge potential  $Z_{(7)}$  that gauges a shift symmetry on the 6-form. Thereby, the proper counting of degrees of freedom will be maintained. The Lagrangian of 11-dimensional supergravity originally given in [9] reads<sup>1</sup>

$$\mathcal{L} = -e R - \frac{1}{12} (e F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} - \frac{1}{216} \varepsilon^{\mu_1 \dots \mu_{11}} F_{\mu_1 \dots \mu_4} F_{\mu_5 \dots \mu_8} A_{\mu_9 \mu_{10} \mu_{11}}) + \mathcal{L}_{\text{fermions}} , \quad (2.1)$$

where the field strength and gauge symmetry of the 3-form are given by

$$F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]} , \quad \delta A_{\mu\nu\rho} = 3\partial_{[\mu} \Lambda_{\nu\rho]} , \quad (2.2)$$

and  $\mathcal{L}_{\text{fermions}}$  represents all terms containing the gravitino.

This action can be reformulated such that it contains a kinetic term for the dual 6-form provided that at the same time the 3-form enters via an additional Chern-Simons-like topological

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<sup>1</sup>We follow the conventions of [9], differing from those of [8]. In particular, we choose the space-time signature to be  $(+ - \dots -)$ , such that  $\varepsilon^{01\dots 11} = \varepsilon_{01\dots 11} = +1$ .

coupling. To be specific, we define the Lagrangian

$$\mathcal{L} = -e R + \frac{2}{7!} e F^{\mu_1 \dots \mu_7} F_{\mu_1 \dots \mu_7} + \mathcal{L}_{\text{top}} + \mathcal{L}_{\text{fermions}} , \quad (2.3)$$

where the topological terms are given by

$$\mathcal{L}_{\text{top}} = -\frac{2}{3 \cdot 7!} \varepsilon^{\mu_1 \dots \mu_{11}} Z_{\mu_1 \dots \mu_7} \partial_{\mu_8} A_{\mu_9 \mu_{10} \mu_{11}} + \frac{1}{12 \cdot 216} \varepsilon^{\mu_1 \dots \mu_{11}} F_{\mu_1 \dots \mu_4} F_{\mu_5 \dots \mu_8} A_{\mu_9 \mu_{10} \mu_{11}} . \quad (2.4)$$

Here, we have defined the field strength of the 6-form as follows

$$F_{\mu_1 \dots \mu_7} = 7 \partial_{[\mu_1} A_{\mu_2 \dots \mu_7]} + Z_{\mu_1 \dots \mu_7} , \quad (2.5)$$

such that it is invariant under the local Stückelberg shift symmetry

$$\delta A_{\mu_1 \dots \mu_6} = -\Sigma_{\mu_1 \dots \mu_6} , \quad \delta Z_{\mu_1 \dots \mu_7} = 7 \partial_{[\mu_1} \Sigma_{\mu_2 \dots \mu_7]} . \quad (2.6)$$

The newly introduced 7-form  $Z_{(7)}$  acts as the shift gauge field.

The theory defined by (2.3) is on-shell equivalent to the original action (2.1). The easiest way to see this is to use the local Stückelberg symmetry (2.6) to gauge-fix  $A_{(6)}$  to zero and then to integrate out  $Z$ . By virtue of the topological term (2.4) this results in the proper kinetic term for the original 3-form in (2.1). Note that this equivalence is not affected by the precise form of the fermionic couplings and therefore supersymmetry extends to (2.3), whose realization we will discuss in sec. 2.3 in more detail.

At this stage one may wonder whether it is not artificial to introduce the 6-form together with a local shift symmetry such that it can be gauged away completely. However, apart from the fact that it needs to be possible to eliminate  $A_{(6)}$  in order to guarantee the equivalence to the original formulation without a 6-form, it is precisely this framework that allows us to analyze the most general gauge symmetries on the 6-form. Since a shift symmetry is the largest possible gauge symmetry, any other gauge invariance consistent with the dynamics of 11-dimensional supergravity has to result from (2.6) by a gauge-fixing. In particular, we will show how the  $E_{11}$  structure indeed arises through a gauge-fixing of (2.6) that is different from gauging  $A_{(6)}$  away.

To start with, we note that the reformulation (2.3) preserves feature (i), i.e. the field equations for  $Z_{(7)}$  and  $A_{(3)}$  are given by the following two first-order ‘duality relations’

$$F_{\mu_1 \dots \mu_7} = \frac{1}{4!} \varepsilon_{\mu_1 \dots \mu_7 \lambda_1 \dots \lambda_4} F^{\lambda_1 \dots \lambda_4} , \quad (2.7)$$

and

$$\partial_{[\mu_1} Z_{\mu_2 \dots \mu_8]} = \frac{35}{4} F_{[\mu_1 \dots \mu_4} F_{\mu_5 \dots \mu_8]} + \text{fermions} . \quad (2.8)$$

The second-order field equation of  $A_{(6)}$  can be obtained from (2.7) by acting with a derivative. Similarly, the second-order equations for  $A_{(3)}$  corresponding to the original action (2.1) can be obtained by first taking the exterior derivative of (2.7) and using the Bianchi identity

$$\partial_{[\mu_1} F_{\mu_2 \dots \mu_8]} = \partial_{[\mu_1} Z_{\mu_2 \dots \mu_8]} , \quad (2.9)$$

and next applying the second duality relation (2.8). Thus, the second-order field equations for the p-form sector of 11-dimensional supergravity can be obtained as the integrability conditions

of a set of two first-order ‘duality’ relations. As we will see below, this is a rather general feature that also holds for the gravity sector.

We now show how in the present case the relation to  $E_{11}$  emerges by a particular gauge-fixing. We first write the right-hand side of (2.8) as the exterior derivative of  $A_{(3)} \wedge F_{(4)}$ , where we momentarily ignore fermionic terms. Consequently, (2.8) can be locally solved by virtue of the Poincaré lemma, implying

$$Z_{\mu_1 \dots \mu_7} = 35 A_{[\mu_1 \dots \mu_3} F_{\mu_4 \dots \mu_7]} + \partial_{[\mu_1} \Xi_{\mu_2 \dots \mu_7]} . \quad (2.10)$$

Here, a new 6-form  $\Xi$  arises, to which we have to assign the following non-trivial gauge transformation in order for  $Z$  to transform as required by (2.6):

$$\delta \Xi_{\mu_1 \dots \mu_6} = 7 \Sigma_{\mu_1 \dots \mu_6} + 42 \partial_{[\mu_1} \Lambda_{\mu_2 \dots \mu_6]} - 105 \Lambda_{[\mu_1 \mu_2} F_{\mu_3 \dots \mu_6]} . \quad (2.11)$$

Here  $\Lambda_{(5)}$  is a new gauge parameter leaving (2.10) invariant. Since  $\Xi$  transforms by a shift under the Stückelberg symmetry, we can gauge-fix this symmetry by setting  $\Xi = 0$ . This in turn requires compensating gauge transformations with parameter

$$\Sigma_{\mu_1 \dots \mu_6} = -6 \partial_{[\mu_1} \Lambda_{\mu_2 \dots \mu_6]} + 15 \Lambda_{[\mu_1 \mu_2} F_{\mu_3 \dots \mu_6]} , \quad (2.12)$$

leaving the following gauge symmetry on  $A_{(6)}$  as the remnant of the shift symmetry (2.6)

$$\delta A_{\mu_1 \dots \mu_6} = 6 \partial_{[\mu_1} \Lambda_{\mu_2 \dots \mu_6]} - 15 \Lambda_{[\mu_1 \mu_2} F_{\mu_3 \dots \mu_6]} . \quad (2.13)$$

Upon redefining the parameter  $\Lambda_{(5)}$ , this transformation rule can be brought into a form in which the symmetry parameters appear only under a derivative,

$$\delta A_{\mu_1 \dots \mu_6} = 6 \left( \partial_{[\mu_1} \Lambda_{\mu_2 \dots \mu_6]} + 10 \partial_{[\mu_1} \Lambda_{\mu_2 \mu_3} A_{\mu_4 \dots \mu_6]} \right) , \quad (2.14)$$

which reproduces the  $E_{11}$  structure (1.1) by choosing

$$\Lambda_{\mu\nu} = \frac{1}{3} \Lambda_{\mu\nu\rho} x^\rho , \quad \Lambda_{\mu_1 \dots \mu_5} = \frac{1}{6} \Lambda_{\rho\mu_1 \dots \mu_5} x^\rho . \quad (2.15)$$

Moreover, after insertion of (2.10), the 7-form field strength reduces to

$$F_{\mu_1 \dots \mu_7} = 7 \partial_{[\mu_1} A_{\mu_2 \dots \mu_7]} + 35 A_{[\mu_1 \dots \mu_3} F_{\mu_4 \dots \mu_7]} , \quad (2.16)$$

which is invariant under (2.13) and corresponds to a Maurer-Cartan form of the non-linear realization of  $E_{11}$ .<sup>2</sup> After this gauge-fixing the first duality relation (2.7) already encodes the full dynamics since the right-hand side of the Bianchi identity (2.9) gets replaced by the right-hand side of (2.8). Upon including the fermions in this analysis it is still possible to solve for  $Z$  since  $A_{(3)}$  couples only via the gauge-invariant field-strength  $F_{(4)}$  to the fermions. As a consequence, the field strengths in (2.7) will be replaced by supercovariant curvatures  $\hat{F}$ , in agreement with supersymmetry. It is this formulation in terms of a single duality relation which is usually presented in order to relate the p-form gauge symmetries of supergravity to

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<sup>2</sup>There exists a slightly different way of presenting the action, in which the  $E_{11}$  structure appears already at the level of the action before gauge-fixing. For this one needs to redefine  $Z$  by a term  $A_{(3)} dA_{(3)}$  in such a way that the Chern-Simons-like structure appears inside the 7-form field strength and not as the usual topological term in the action. The latter is then generated upon integrating out  $Z$ .

$E_{11}$ . However, this procedure leading to a single duality relation cannot be implemented at the level of the action, since derivatives were involved when solving some of the field equations. Therefore, the field  $Z$  is indispensable in order to obtain the first-order duality equations from an action. We will now discuss a similar reformulation with Stückelberg gauge fields that involves dual gravity.

## 2.2 Democratic formulation with dual graviton

In order to introduce the dual graviton, it is convenient to use that the Einstein-Hilbert action in  $D = 11$  can be written, up to total derivatives, in the following first-order form [1, 12]

$$\mathcal{L}_{\text{EH}} = \frac{1}{2}e \left( Y^{ab|c} \Omega_{abc} - \frac{1}{2} Y_{ab|c} Y^{ac|b} + \frac{1}{18} Y_{ab|}{}^b Y^{ac|}{}_c \right), \quad (2.17)$$

where

$$\Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu (\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c) \quad (2.18)$$

are the coefficients of anholonomy, and  $Y_{ab|c} = -Y_{ba|c}$  is an auxiliary field with no further symmetry properties, i.e., transforming in a reducible representation. Integrating out  $Y$  one recovers the Einstein-Hilbert action written through  $\Omega^2$  terms. Below we will use that any symmetry of the original Einstein-Hilbert action can be extended to an (off-shell) invariance of the first-order action (2.17) by assigning a suitable transformation rule to  $Y_{ab|c}$ . More precisely, an arbitrary symmetry of the Einstein-Hilbert action with transformations  $\delta e_\mu{}^a$  is elevated to a symmetry of (2.17) with  $\delta Y_{ab|c}$  given by

$$\delta Y_{ab|c} = \delta \Omega_{abc} - 2\delta \Omega_{c[ab]} + 4\eta_{c[a} \delta \Omega_{b]d}{}^d. \quad (2.19)$$

We next define the Hodge dual via

$$Y^{ab|c} = \frac{1}{9!} \epsilon^{abc_1 \dots c_9} Y_{c_1 \dots c_9|}{}^c, \quad (2.20)$$

which yields a  $(9, 1)$  tensor and which will below play the same role as the 7-form shift gauge field  $Z$  in the previous subsection. We now consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_C(e, G) + \frac{2}{7!} e F^{\mu_1 \dots \mu_7} F_{\mu_1 \dots \mu_7} + \widehat{\mathcal{L}}_{\text{top}} + \mathcal{L}_{\text{fermions}}, \quad (2.21)$$

where  $\mathcal{L}_C(e, G)$  is in form equal to the so-called Curtright Lagrangian for the dual graviton  $C_{\mu_1 \dots \mu_8}{}^a$ ,

$$\begin{aligned} \mathcal{L}_C(e, G) = & -\frac{1}{9!} \left( \frac{2}{9} e G^{\mu_1 \dots \mu_9 a} G_{\mu_1 \dots \mu_9 a} - \frac{9}{4} e e_\nu{}^a e_b{}^\rho G^{\mu_1 \dots \mu_8 \nu}{}_a G_{\mu_1 \dots \mu_8 \rho}{}^b \right. \\ & \left. + \frac{1}{4} e e_\nu{}^b e_a{}^\rho G^{\mu_1 \dots \mu_8 \nu a} G_{\mu_1 \dots \mu_8 \rho b} \right). \end{aligned} \quad (2.22)$$

(We will discuss in sec. 3 why it is justified to call  $C$  the ‘dual graviton’.) To be precise, while the conventional Curtright action [10, 11] for the dual graviton is formulated on flat space, here we keep the dynamical metric in order to maintain full diffeomorphism invariance. Another difference with the Curtright action is that the field strength entering here is the shift-invariant combination

$$G_{\mu_1 \dots \mu_9}{}^a = 9 \partial_{[\mu_1} C_{\mu_2 \dots \mu_9]}{}^a + Y_{\mu_1 \dots \mu_9}{}^a, \quad (2.23)$$

admitting the symmetry

$$\delta Y_{\mu_1 \dots \mu_9}{}^a = 9 \partial_{[\mu_1} \Sigma_{\mu_2 \dots \mu_9]}{}^a, \quad \delta C_{\mu_1 \dots \mu_8}{}^a = -\Sigma_{\mu_1 \dots \mu_8}{}^a. \quad (2.24)$$

Moreover, the topological couplings in (2.4) have been extended by a term involving the original vielbein  $e_\mu{}^a$ ,

$$\widehat{\mathcal{L}}_{\text{top}} = \mathcal{L}_{\text{top}} + \frac{1}{9!} \varepsilon^{\mu_1 \dots \mu_9 \nu \rho} Y_{\mu_1 \dots \mu_9 a} \partial_\nu e_\rho{}^a. \quad (2.25)$$

We are working here in a frame-like formulation (with flat indices  $a, b, \dots$ ) for which the dual graviton lives in a reducible representation. The antisymmetric indices are chosen to be curved, while the extra index is flat. This assignment is natural in that it keeps the diffeomorphism symmetry manifest, leaving the local Lorentz group as the only non-manifest symmetry [8].

It can now be shown in complete analogy to the discussion in the previous subsection that the action (2.21) containing all fields required by  $E_{11}$  is on-shell equivalent to 11-dimensional supergravity. To show this equivalence one may gauge-fix the dual graviton to zero, after which the terms containing  $Y$  are given by (2.17) when rewritten according to (2.20). Thus, it is equivalent to the Einstein-Hilbert action. The analysis of the foregoing section concerning the 3-form/6-form sector is unaffected, and so (2.21) is equivalent to 11-dimensional supergravity.

Let us now inspect the equations of motion. Varying with respect to  $Y$  we obtain a duality relation between the original vielbein and the dual graviton,

$$e^{-1} \varepsilon^{\mu_1 \dots \mu_9 \nu \rho} \Omega_{\nu \rho}{}^a = \frac{8}{9} G^{\mu_1 \dots \mu_9 a} - 9 e_{\rho b} e^{a[\mu_1} G^{\mu_2 \dots \mu_9] \rho b} + e_\rho{}^a e_b{}^{[\mu_1} G^{\mu_2 \dots \mu_9] \rho b}, \quad (2.26)$$

while variation with respect to the vielbein yields

$$-\frac{1}{9!} e^{-1} \varepsilon^{\mu \mu_1 \dots \mu_{10}} \partial_{\mu_1} Y_{\mu_2 \dots \mu_{10} a} = e^{-1} \mathcal{T}^\mu{}_a \equiv e^{-1} \frac{\delta \mathcal{L}_C(e, G)}{\delta e_\mu{}^a} + e^{-1} T^\mu{}_a, \quad (2.27)$$

where  $T^\mu{}_a$  denotes the energy-momentum tensor of  $A_{(6)}$ . As in the 3-form/6-form example, the second-order field equations for the dual graviton  $C$  can be obtained from (2.26) by taking the exterior derivative. Therefore, the full set of field equations including the non-linear Einstein equations is encoded in the two first-order ‘duality’ relations (2.26) and (2.27), preserving feature (i), cf. the introduction, at the non-linear level. In particular, this circumvents the problem that it is not possible to ‘pull out’ a derivative of the energy-momentum tensor and that it is, therefore, not possible to encode matter couplings in a single duality relation [6]. The way out is to introduce a new gauge field  $Y$  together with a second duality relation. We stress that the need of having *two* duality relations is by no means a peculiarity of gravity. For instance, for obtaining the scalar field equations in gauged supergravity from first-order duality relations one is confronted with the analogous problem that it is not possible to ‘pull out’ a derivative from the source term induced by the scalar potential. The resolution is, again, to introduce a second duality relation involving a higher-rank p-form [13], in accordance with the so-called tensor hierarchy [7].

## 2.3 Gauge symmetries

In this subsection we are going to show how in the proposed reformulation of  $D = 11$  supergravity the symmetries of the original 11-dimensional supergravity theory, as for instance

supersymmetry, are realized. Due to the on-shell equivalence of the two formulations, the existence of these symmetries in a local form is guaranteed. It is, however, instructive to determine them explicitly.<sup>3</sup>

We focus first on the p-form sector only, i.e., we assume that in the gravitational sector only the ordinary metric enters, via the standard Einstein-Hilbert action. The following discussion shows how any symmetry of 11-dimensional supergravity can be elevated to a symmetry of the reformulation. We first note that the variation of the original kinetic term for the 3-form reads

$$\begin{aligned} \delta \mathcal{L}_{3\text{-form}} &= \frac{2}{3} \delta A_{\mu_1 \dots \mu_3} \partial_\rho F^{\rho \mu_1 \dots \mu_3} \\ &\quad - \frac{1}{12} \delta g^{\mu\nu} \left( 4\sqrt{g} F_\mu^{\rho_1 \dots \rho_3} F_{\nu \rho_1 \dots \rho_3} - \frac{1}{2} \sqrt{g} g_{\mu\nu} F^{\mu_1 \dots \mu_4} F_{\mu_1 \dots \mu_4} \right) . \end{aligned} \quad (2.28)$$

In the case of supersymmetry, the variations of the elf-bein and the 3-form are given by

$$\delta e_\mu^a = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu , \quad \delta A_{\mu_1 \dots \mu_3} = \frac{3}{2} \bar{\epsilon} \gamma_{[\mu_1 \mu_2} \psi_{\mu_3]} . \quad (2.29)$$

One may verify that the variation of the kinetic term for the 6-form and the additional topological term containing  $Z$  precisely reproduces (2.28) provided we assign the following transformation rules to the new fields

$$\delta Z_{\mu_1 \dots \mu_7} = \frac{1}{4!} \varepsilon_{\mu_1 \dots \mu_7 \nu_1 \dots \nu_4} \delta F^{\nu_1 \dots \nu_4} + \frac{1}{4} g_{\mu\nu} \delta g^{\mu\nu} \mathcal{E}_{\mu_1 \dots \mu_7}^+ + \frac{7}{2} \delta g_{\nu[\mu_1} \mathcal{E}_{+\mu_2 \dots \mu_7]}^\nu , \quad (2.30)$$

$$\delta A_{\mu_1 \dots \mu_6} = 0 , \quad (2.31)$$

while the old ones remain unmodified. Here we have defined

$$\mathcal{E}_\pm^{\mu_1 \dots \mu_7} = F^{\mu_1 \dots \mu_7} \pm \frac{1}{4!} \varepsilon^{\mu_1 \dots \mu_7 \nu_1 \dots \nu_4} F_{\nu_1 \dots \nu_4} , \quad (2.32)$$

of which the  $\mathcal{E}_-$  combination represents the duality relation, i.e., vanishes on-shell,  $\mathcal{E}_- = 0$ , while the other combination is on-shell given by the field strength,  $\mathcal{E}_+ = 2F_{(7)}$ . It is amusing to note that in this sense precisely the ‘opposite’ of the duality relation enters in (2.30). In total we have shown that (2.3) is invariant under the combined transformation of the old fields and (2.30). The fact that the symmetry transformation on  $A_{(6)}$  can be taken to vanish is due to the Stückelberg invariance. Once the latter is gauge-fixed as in sec. 2.1, compensating gauge transformations are required, which in turn give rise to non-trivial transformations of  $A_{(6)}$ . In the case of supersymmetry, this procedure leads to the following supersymmetry rule:

$$\delta A_{\mu_1 \dots \mu_6} = 3 \bar{\epsilon} \gamma_{[\mu_1 \dots \mu_5} \psi_{\mu_6]} , \quad (2.33)$$

where we ignore higher-order terms. Here we used the field equations, i.e., this supersymmetry rule holds on-shell.

Similar conclusions apply to the gravitational sector once the dual graviton is introduced in the Lagrangian (2.21). In fact, in (2.19) we have already given the off-shell symmetry relevant for this reformulation, which determines the symmetry rules for the shift gauge field  $Y$ . Due to the Stückelberg invariance, the dual graviton can be taken to be invariant, and it will only start transforming after a gauge-fixing.

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<sup>3</sup>In the context of  $N = 1$  supergravity in  $D = 4$  the supersymmetry of this reformulation has been analyzed in [14], however, with inconclusive results.



### 3 Can the gauge symmetries be reconciled with $E_{11}$ ?

So far we have shown that it is possible to reformulate the action of 11-dimensional supergravity in such a way that it contains the fields required by  $E_{11}$  at low levels together with two extra Stückelberg gauge fields. Moreover, the field equations can be encoded in a set of first-order ‘duality’ relations, in agreement with feature (i) mentioned in the introduction. In this section, we are going to investigate to what extent this realizes also the  $E_{11}$  symmetry, i.e., whether at the same time the reformulation preserves feature (ii).

We first discuss in which sense it is justified to call  $C$  the ‘dual graviton’. Even though  $C$  can be gauged away at any step, there is a limit in which there is a different gauge-fixing, giving rise to a propagating dual graviton. More precisely, in the linearization about flat space gravity decouples from matter and thus the second duality relation (2.27) reduces to  $dY^a = 0$ . Since so far  $Y^a$  does not transform in a specific Young tableaux representation, the index  $a$  can be treated as a redundant index. Thus, applying the Poincaré lemma, the shift gauge field is determined to be pure gauge,

$$\delta Y_{\mu_1 \dots \mu_9}{}^a = 9 \partial_{[\mu_1} \Xi_{\mu_2 \dots \mu_9]}{}^a, \quad \delta \Xi_{\mu_1 \dots \mu_8}{}^a = \partial_{[\mu_1} \gamma_{\mu_2 \dots \mu_8]}{}^a + \Sigma_{\mu_1 \dots \mu_8}{}^a, \quad (3.1)$$

and can therefore be gauge-fixed to zero. By (3.1) this requires compensating gauge transformations on the dual graviton, giving rise to

$$\delta_\gamma C_{\mu_1 \dots \mu_8}{}^a = \partial_{[\mu_1} \gamma_{\mu_2 \dots \mu_8]}{}^a. \quad (3.2)$$

These are the ‘dual diffeomorphisms’, while the action reduces to the Curtright action for the dual graviton, being invariant under (3.2). To be more precise, the theory still has a local Lorentz symmetry which makes it possible to gauge away the totally antisymmetric part of  $C$ , leaving the dual graviton in the (8,1) Young tableaux representation [8]. Correspondingly, (3.2) reduces to a (non-manifest) gauge symmetry with two parameters in irreducible representations,<sup>4</sup>

$$\delta C_{\mu_1 \dots \mu_8, \nu} = \partial_{[\mu_1} \alpha_{\mu_2 \dots \mu_8], \nu} + \partial_{(\mu_1} \beta_{\mu_2 \dots \mu_8 \nu)}, \quad (3.3)$$

where  $\alpha$  transforms in the (7,1) tableaux and  $\beta$  is fully antisymmetric. This gauge symmetry of the dual graviton can be associated to the global shift transformation predicted by non-linear realizations of Kac-Moody algebras. More precisely, choosing

$$\alpha_{\mu_1 \dots \mu_7, \nu} = \xi_{\mu_1 \dots \mu_7 \rho, \nu} x^\rho \quad \text{or} \quad \beta_{\mu_1 \dots \mu_8} = \xi_{\mu_1 \dots \mu_8, \nu} x^\nu, \quad (3.4)$$

one recovers the global symmetry transformation encoded in the first term of (1.2), which is precisely the symmetry predicted for pure 11-dimensional gravity based on the Kac-Moody algebra  $A_8^{+++}$ .

We now turn to the question whether also the  $E_{11}$  structure going beyond the dual diffeomorphisms, i.e. the remaining terms in the transformation rule (1.2), can be obtained in this way. First we observe that due to the Stückelberg symmetry (2.24) *any* symmetry can be realized on the dual graviton by simply choosing the shift parameter  $\Sigma$  in the required way, as

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<sup>4</sup>In (3.3) we use that on flat space there is no distinction between flat and curved indices.

for instance suggested by the  $E_{11}$  structure (1.2). However, this trivial way of realizing  $E_{11}$  is clearly unsatisfactory. As in our previous discussions, what we really have to ask for is a gauge-fixing that allows to eliminate the Stückelberg gauge field  $Y$  in such a way that the residual gauge symmetry on the dual graviton is given by (1.2). For this to be the case, it has to be possible to solve the second duality relation (2.27) for  $Y$  up to pure gauge degrees of freedom. As the two terms appearing on the right-hand side of that duality relation can be interpreted as the energy-momentum tensor of the dual graviton and the 6-form, respectively, this problem is similar to the one encountered in [6] of pulling out a derivative of the energy-momentum tensor, which turned out to be impossible. Even though here the situation is slightly different in that we are not dealing with the ordinary energy-momentum tensor of 11-dimensional supergravity but instead with the tensor  $\mathcal{T}^\mu_a$ , see eq. (2.27), involving the dual graviton and 6-form together with their respective shift gauge fields, it is not possible to find a local expression for  $Y^a$  that solves (2.27). To show this one may gauge-fix  $C$  and  $A_{(6)}$  away, after which this tensor symbolically reads  $\mathcal{T} \sim Y^2 + Z^2$ , such that one is left with a  $Z^2$  term that cannot be written as the derivative of some local expression.

We conclude that, unlike the 3-form/6-form sector, the dual gravity sector does not allow the elimination of the shift gauge field  $Y$ . It is therefore impossible to generate the transformation rule (1.2) predicted by  $E_{11}$  for the dual graviton as the result of a compensating gauge transformation in the absence of  $Y$ .

## 4 Conclusions

In this letter we addressed the problem of reconciling the dual graviton with  $E_{11}$ . For this we used the recently proposed reformulation of gravitational theories [8], which involves the dual graviton and circumvents the no-go results of [5, 6] by virtue of keeping the original graviton via a topological term in such a way that upon linearization the spin-2 degrees of freedom can be encoded either in the graviton or its dual. This reformulation contains all fields required by  $E_{11}$  (up to the given level) and allows to encode the field equations in terms of an enlarged set of first-order ‘duality relations’. Though property (i) discussed in the introduction is therefore maintained, in contrast to the p-form sector this is a priori independent of the validity of requirement (ii) according to which this reformulation should realize the  $E_{11}$  symmetry on the dual fields. In answering the question whether (a truncation of)  $E_{11}$  is a symmetry of 11-dimensional supergravity, a crucial role is played by the local shift symmetry that is realized on the dual fields. As a consequence of this local shift symmetry the dual fields capture the most general symmetries in that any supposed gauge invariance of 11-dimensional supergravity has to result from this shift symmetry by a gauge-fixing. The question whether 11-dimensional supergravity does or does not non-trivially realize the symmetry (1.2) on the dual graviton predicted by  $E_{11}$  can therefore be made precise by asking whether there is a gauge-fixing that eliminates the shift gauge field such that the residual gauge invariance gives rise to the symmetry rule predicted by  $E_{11}$ . We find that this is not possible. In this sense  $E_{11}$  by itself is not a symmetry of 11-dimensional supergravity, but can at most be part of an extended symmetry structure going beyond  $E_{11}$  and comprising the additional Stückelberg fields.

It is interesting to compare our results with a recent exploration of the way in which the local symmetries of supergravity can be obtained from the global symmetries of non-linear realisations of  $E_{11}$  [15]. In [15] the transition from global to local symmetries was implemented by introducing on top of each  $E_{11}$  generator in the Borel subalgebra an infinite set of so-called Ogievetsky generators. In the case of p-forms the fields associated to these Ogievetsky generators parameterize higher derivatives of the p-form fields, thus extending the global symmetry to a local one without introducing new fields. This situation changes for the mixed symmetry generators like the one corresponding to the dual graviton considered in this letter. In that case not all Ogievetsky generators can be eliminated. In particular, one is left with a curvature for the dual graviton, linear in derivatives, that contains a ‘shift’ gauge field corresponding to one of the Ogievetsky generators. This resonates with our approach. Our results therefore suggest that all fields associated to the Ogievetsky generators, except for the shift gauge field  $Y$  occurring in the dual graviton curvature  $G$ , can be expressed in terms of derivatives of the basic dual graviton curvature. Even though it seems to be difficult to reconcile the extended symmetry structure of [15] with the original  $E_{11}$  beyond the Borel subalgebra, it might be worth to investigate it in light of the present model and its possible relation to supergravity and/or M-theoretic extensions.

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